Partial Wave Analysis of Nucleon-Nucleon Scattering below pion production threshold*

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We undertake a simultaneous partial wave analysis to proton-proton and neutron-proton scattering data below pion production threshold up to LAB energies of 350MeV. We represent the interaction as a sum of delta-shells in configuration space below the 3fm and a charge dependent one pion exchange potential above 3fm together with electromagnetic effects. We obtain $\chi^2 = 2740|_{pp} + 3835|_{np}$ with a total of $N = 2734|_{pp} + 3610|_{np}$ data obtained till 2007 and a total number of 46 fitting parameters yielding $\chi^2/\text{d.o.f} = 1.04$. Special attention is payed to estimate the errors of the phenomenological interaction as well as the derived effects on the phase-shifts and scattering amplitudes.

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The NN interaction plays a central role in Nuclear Physics (see e.g. [1, 2] and references therein). The standard procedure to constrain the interaction uses a partial wave analysis (PWA) of the proton-proton (pp) and neutron-proton (np) scattering data below pion production threshold [3]. The best PWA uses a large body of NN scattering data giving $\chi^2/\text{d.o.f} \lesssim 1$ after discarding about 20% of 3σ inconsistent data [4]. This fit incorporates charge dependence (CD) for the One Pion Exchange (OPE) potential as well as electromagnetic, vacuum polarization and relativistic effects, the latter being key ingredients to this accurate success. The analysis was more conveniently carried out using an energy dependent potential for the short range part. Later on energy independent *high quality* potentials were designed with almost identical $\chi^2/\text{d.o.f} \sim 1$ for the gradually increasing database [5–8]. While any of these potentials provides individually satisfactory fits to the available experimental data, an error analysis would add a means of estimating quantitatively the impact of NN-scattering uncertainties in Nuclear Structure calculations. In the present work we provide a new high-quality potential implementing an analysis of its parameter uncertainties using the standard method of inverting the covariance matrix [9].

The complete on-shell NN scattering amplitude contains five independent complex quantities, which we choose for definiteness as the Wolfenstein parameters [3]

$$M(\mathbf{k}_f,\mathbf{k}_i) = a + m(\sigma_1,\mathbf{n})(\sigma_2,\mathbf{n}) + (g-h)(\sigma_1,\mathbf{m})(\sigma_2,\mathbf{m}) + (g+h)(\sigma_1,\mathbf{l})(\sigma_2,\mathbf{l}) + c(\sigma_1+\sigma_2,n), \qquad (1)$$

where a,m,g,h,c depend on energy and angle. To determine these parameters and their uncertainties we find that a convenient representation to sample the short distance contributions to the NN interaction can be written as a sum of delta-shells

$$V(r) = \sum_{n=1}^{18} O_n \left[\sum_{i=1}^{N} V_{i,n} r_i \delta(r-r_i) \right] + \Big[V_{\rm OPE}(r) + V_{\rm em}(r) \Big] \theta(r-r_c), \qquad (2)$$

where O_n are the set of operators in the AV18 basis [6], $r_i \le r_c$ are a discrete set of N-radii and $V_{i,n}$ are unknown coefficients to be determined from data. The $r > r_c$ piece contains a CD OPE and electromagnetic (EM) corrections which is kept fixed throughout. The solution of the corresponding Schrödinger equation in the (coupled) partial waves $^{2S+1}L_J$ for $r \le r_c$ is straightforward since the potential reads

$$V^{JS}_{l,l'}(r) = \frac{1}{2\mu_{\alpha\beta}} \sum_{i=1}^{N} (\lambda_i)^{JS}_{l,l'} \delta(r-r_i) \qquad r \le r_c \qquad (3)$$

with $\mu_{\alpha\beta} = M_\alpha M_\beta/(M_\alpha+M_\beta)$ the reduced mass with $\alpha,\beta = n,p$. Here, $(\lambda_i)^{JS}_{l,l'}$ are related to the $V_{i,n}$ coefficients by linear transformation at each discrete point r_i. Thus, for any $r_i < r < r_{i+1}$ we have free particle solutions and log-derivatives are discontinuous at the r_i-points so that one generates an accumulated S-matrix at any sampling point providing a discrete and purely algebraic version of Calogero's variable phase equation [16].

This form of potential effectively implements a coarse graining of the interaction, first proposed 40 years ago by Aviles [17]. We have found that the representation (3) is extremely convenient and computationally cheap for our PWA. The low energy expansion of the discrete variable phase equations was used already in Ref. [18] to determine threshold parameters in all partial waves. The relation to the well-known Nyquist theorem of sampling a signal with a given bandwidth has been discussed in Ref. [19]. Some of the advantages of directly using this simple potential for Nuclear Structure calculations have also been analyzed [20].

The fact that we are coarse graining the interaction enables to encode efficiently all effects operating below the finest

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TABLE I. Fitting delta-shell parameters $(\lambda_n)^{JS}_{l,l'}$ (in fm^{-1}) with their errors for all states in the JS channel. We take $N=5$ equidistant points with $\Delta r = 0.6$fm. $-$ indicates that the corresponding $(\lambda_n)^{JS}_{l,l'}=0$. In the first line we provide the central component of the delta shells corresponding to the EM effects below $r_c = 3$fm. These parameters remain fixed within the fitting process.

Wave	λ_1 ($r_1=0.6$fm)	λ_2 ($r_2=1.2$fm)	λ_3 ($r_3=1.8$fm)	λ_4 ($r_4=2.4$fm)	λ_5 ($r_5=3.0$fm)
$V_C[\mathrm{pp}]_{\mathrm{EM}}$	0.02091441	0.01816750	0.00952244	0.01052224	0.00263887
$^1S_0$[pp]	1.32(2)	-0.720(4)	-0.189(2)	–	-0.0215(3)
$^1S_0$[np]	1.37(7)	-0.77(2)	-0.17(1)	–	-0.025(1)
$^3P_0$	–	1.11(2)	-0.371(7)	-0.041(3)	-0.030(1)
$^1P_1$	–	1.27(2)	–	0.069(2)	–
$^3P_1$	–	1.379(5)	–	0.0569(5)	–
$^3S_1$	1.52(6)	-0.44(1)	–	-0.061(1)	–
ε_1	–	-1.64(1)	-0.38(2)	-0.206(8)	-0.024(3)
$^3D_1$	–	–	0.38(1)	0.081(9)	0.016(3)
$^1D_2$	–	-0.23(1)	-0.200(3)	–	-0.0198(2)
$^3D_2$	–	-0.92(4)	-0.23(2)	-0.210(8)	-0.019(2)
$^3P_2$	–	-0.485(1)	–	-0.0241(6)	-0.0061(3)
ε_2	–	0.25(2)	0.212(4)	0.040(2)	0.0162(5)
$^3F_2$	–	3.47(6)	-0.234(4)	–	-0.0139(6)
$^1F_3$	–	–	0.17(2)	0.071(7)	–
$^3D_3$	–	0.55(2)	–	–	–

TABLE II. Deuteron static properties compared with empirical values and high-quality potentials calculations

	Delta Shell	Empirical[10–15]	Nijm I [5]	Nijm II [5]	Reid93 [5]	AV18 [6]	CD-Bonn [7]
E_d(MeV)	Input	2.224575(9)	Input	Input	Input	Input	Input
η	0.02493(8)	0.0256(5)	0.02534	0.02521	0.02514	0.0250	0.0256
$A_S(\mathrm{fm}^{1/2})$	0.8829(4)	0.8781(44)	0.8841	0.8845	0.8853	0.8850	0.8846
r_m(fm)	1.9645(9)	1.953(3)	1.9666	1.9675	1.9686	1.967	1.966
$Q_D(\mathrm{fm}^2)$	0.2679(9)	0.2859(3)	0.2719	0.2707	0.2703	0.270	0.270
P_D	5.62(5)	5.67(4)	5.664	5.635	5.699	5.76	4.85
$\langle r^{-1}\rangle(\mathrm{fm}^{-1})$	0.4540(5)			0.4502	0.4515		

resolution Δr which corresponds to the shortest de Broglie wavelength corresponding to the pion production threshold, $\lambda_{\min} \sim 1/\sqrt{m_\pi M_N} \sim 0.55$fm, so that a maximal number of points $N = r_c/\Delta r \sim 5$ (for $r_c = 3$fm) should be needed. In practice, we expect the number of sampling points to decrease with angular momentum as the centrifugal barrier make irrelevant those points $r_i \lesssim (l+1/2)/p$ below the relevant impact parameter, so that the total number of integration points and hence fitting strengths V_{in} will be limited and smaller than $N=5$.

The previous discretization of the potential is just a way to numerically solve Schrödinger equation for any given potential where one replaces $V(r) \to V(r) = \sum_i V(r_i) r_i \delta(r-r_i)$, but the number of delta-shells may be quite large for *fixed* strengths $V(r_i)$. For instance, for the $^1S_0$ wave and for the AV18 [6] potential one needs N=600 delta-shells to reproduce the phase-shift with sufficient accuracy (below 10^{-4} degrees) but just $N=5$ if one uses V_i as fitting parameters to the *same* phase shift [20].

The EM part of the NN potential gives a contribution to the scattering amplitude that must be taken into account properly in order to correctly calculate the different observables. Each term of the electromagnetic potential in the pp and np channels needs to be treated differently to obtain the corresponding parts of the total EM amplitude. The expresions for the contributions coming from the pp one photon exchange potential V_{C1}, and the corresponding relativistic correction V_{C2}, are well known and can be found in [4]. To calculate the contribution of the the vacuum polarization term V_{VP} we used the approximation to the amplitude given in [21]. Finally, Ref. [22] details the treatment of the magnetic moment interaction V_{MM} for both pp and np channels and the necesary corrections to the nuclear amplitude coming from the electromagnetic phaseshifts.

Of course, once we admit that the interaction below r_c is unknown there is no gain in directly extending the well-

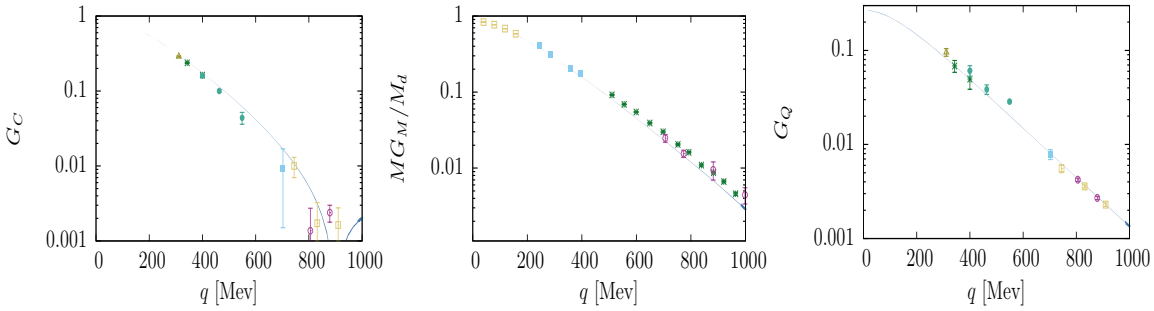

FIG. 1. Deuteron Form Factors with error bands by propagating the uncertainties. Note the tiny width of the bands.

known charge-dependent OPE tail for $r \leq r_c$. Unlike the purely strong piece of the NN potential the electromagnetic contributions are known with much higher accuracy and to shorter distances (see e.g. Ref. [6]) so that one might extend $V_{\rm em}(r)$ below r_c adding a continuous contribution on top of the delta-shells, so that the advantage of having a few points in the region $r \leq r_c$ would be lost. To improve on this we coarse-grain the EM interaction up to the pion production threshold. Thus, we look for a discrete representation on the grid of the purely EM contribution $V_{\rm em}(r)$, i.e. we take $\overline{V}_{\rm em}(r) = \sum_n V_i^C r_i \delta(r-r_i) + \theta(r-r_c) V_{\rm em}(r)$, where the V_i^C are determined by reproducing the purely EM scattering amplitude to high-precision and are not changed in the fitting process. The result using the EM potential of Ref. [6] just turns out to involve the Coulomb contribution in the central channel and is given in the first line of Table I. As expected from Nyquist sampling theorem, we need at most $N = 5$ sampling points which for simplicity are taken to be equidistant with $\Delta r = 0.6$fm between the origin and $r_c = 3$fm to coarse grain the EM interaction below $r \leq r_c$. Thus we should have $V_i^{pp} = V_i^{np} + V_i^C$ if charge symmetry was exact in strong interactions for $r < r_c$, although some corrections are expected as documented below.

In our fitting procedure we coarse grain the unknown short range part of the interaction from the scattering data. We use a large database compiling proton-proton and neutron-proton scattering data obtained till 2007 [23–25] and carry out at any rate a simultaneous pp and np fit. Unfortunately, some groups of these data have a common but unknown normalization. We thus use the standard floating by including an additional contribution to the χ^2 as explained in detail in Ref. [8]. We also apply the Nijmegen PWA [4] 3σ-criterion to reject possible outliers from the main fit with a 3σ-confidence level, a strategy reducing the minimal χ^2 but also enlarging the uncertainties. Initially we consider $N = 2900|_{pp} + 5034|_{np}$ data and get $\chi^2_{\rm min} = 3375|_{pp} + 9217|_{np}$ yielding $\chi^2/{\rm d.o.f.} = 1.59$. Applying the 3σ-rejection we discard 1590 data. After re-fitting the remaining $N = 2734|_{pp} + 3610|_{np}$ data we finally obtain $\chi^2_{\rm min} = 2740|_{pp} + 3835|_{np}$ with a total number of 46 fitting parameters so that $\chi^2/{\rm d.o.f.} = 1.04$.

While the linear relations of the $(\lambda_i)^{JS}_{l,l'}$ and $V_{i,n}$ parameters are straightforward, limiting the number of operators O_n reduces the number of independent partial waves. The fitting parameters $(\lambda_n)^{JS}_{l,l'}$ entering the delta-shell potentials as independent variables, Eq. (3), are listed in Table I with their deduced uncertainties. All other partial waves are consistently obtained from those. Our final results allow to fix the *same* pp and np potential parameters with the exception of the central components of the potential as it is usually the case in all joint pp+np analyses carried out so far [4–7].

We find that introducing more points or equivalently reducing Δr generates unnecessary correlations and does not improve the fit. Also, lowering the value of r_c below 3fm, requires overlapping the short-distance potential, Eq. (3), with the OPE plus em corrections. We find that independent fits to pp and np, while reducing each of the χ^2-values, drive the minimum to incompatible parameters and erroneous np phases in isovector channels. Actually, the pp data constrain these channels most efficiently and in a first step pp-fits where carried out to find suitable starting parameters for the corresponding np-phases. Quite generally, we have checked that the minimum is robust by proposing several starting solutions.

As a numerical check of our construction of the amplitudes we reproduced the Wolfenstein parameters for the Reid93 and NijmII potentials to high accuracy using $N = 12000$ delta-shells grid points, which ensures correctness of the strong contributions. As a further check of our implementation of the long-range EM effects along the lines of Refs. [4, 21, 22] we have also computed the χ^2/d.o.f. for Reid93, NijmII and AV18 potentials (fitted to data prior to 1993) which globally and bin-wise are reasonably well reproduced when our database (coinciding with the one of Ref. [8] for np) includes only data prior to 1993.

We determine the deuteron properties by solving the bound state problem in the ${}^3S_1 - {}^3D_1$ channel using the corresponding parameters listed in Table I. The predictions are presented in table II where our quoted errors are obtained from propagating those of Table I by using the full covariance matrix among fitting parameters. The comparison with experimental values or high quality potentials where the deuteron binding energy is used as an input is satisfactory [4–8].

The outcoming and tiny error bands for the Deuteron form factors (see e.g. [26]) are depicted in Fig. 1. The rather small discrepancy between calculated and experimental values is statistically significant and might be resolved by the inclusion of Meson Exchange Currents. In Fig. 2 we show the active pp

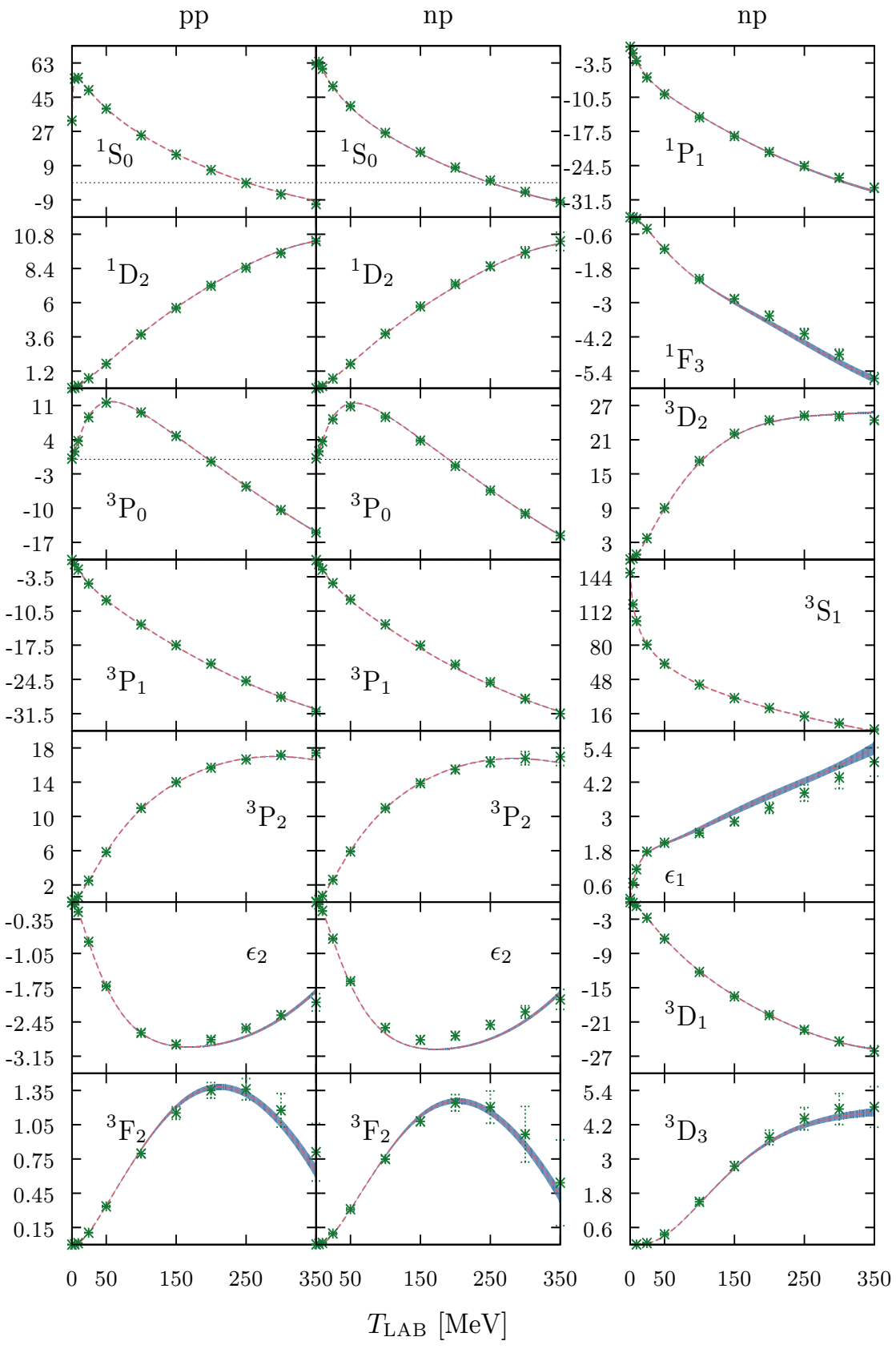

FIG. 2. np and pp phase shifts and their errors (solid band) corresponding to independent operator combinations corresponding to the fitted potential. The points with error bars represent the mean value and standard deviation of the compilation of the PWA [4] and the six high quality potentials [5–8].

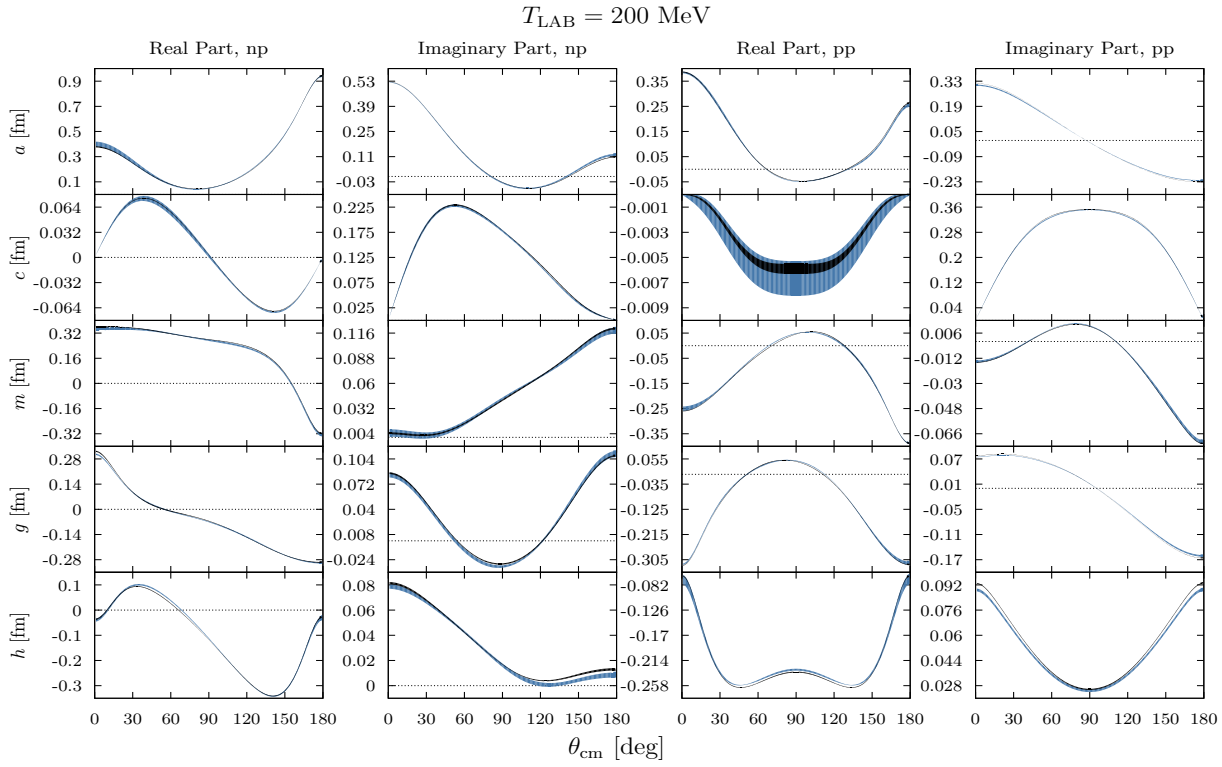

FIG. 3. np (left) and pp (right) Wolfenstein parameters (in fm) as a function of the CM angle (in degrees) and for $E_{\rm LAB} = 200$MeV. We compare our fit (black band) with a compilation (blue band) of the PWA and six high quality potentials [4, 6–8] which provided a $\chi^2/{\rm d.o.f} \lesssim 1$.

and np phases in the fit with their propagated errors and compare them with a compilation of the PWA and six high quality potentials [4, 6–8] which provided a $\chi^2/{\rm d.o.f} \lesssim 1$. Likewise, in Fig. 3 we also show a similar comparison for the pp and np Wolfenstein parameters.

Finally, as the previous analyses [4–8] and the present paper show the form of the potential is not unique providing a source of systematic errors. A first step along these lines has been undertaken in Ref. [27]. Thus, the uncertainties will generally be larger than those of purely statistical nature estimated here.

To summarize, we have determined a high-quality proton-proton and neutron-proton interaction from a simultaneous fit to scattering data and the deuteron binding energy with $\chi^2/{\rm d.o.f.} = 1.04$. Our short range potential consists of a few delta-shells for the lowest partial waves. In addition, charge-dependent electromagnetic interactions and one pion exchange are implemented. We provide error estimates on our fitting parameters. Futher details will be presented elsewhere.

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